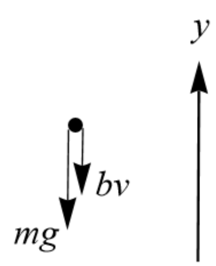
Problem 2.11

Consider an object that is thrown vertically up with initial speed v_o in a linear medium.
(a) Measuring y *upward* from the point of release, write expressions for the object's velocity $v_y(t)$ and position $y(t)$. **(b)** Find the time for the object to reach its highest point and its position y_{max} at that point. **(c)** Show that as the drag coefficient approaches zero, your last answer reduces to the well-known result $y_{max} = \frac{1}{2}v_o^2/g$ for an object in the vacuum. [*Hint:* If the drag force is very small, the terminal speed is very big, so v_o/v_{ter}, is very small. Use the Taylor series for the log function to approximate $\ln(1+\delta)$ by $\delta - \frac{1}{2}\delta^2$. (For a little more on Taylor series see Problem 2.18.)]

Solution

Part (a)

Draw a free body diagram for an object travelling upward, assuming there's only linear air resistance.



Apply Newton's second law in the y-direction to get the equation of motion, letting $v_y = v$.

$$\sum F_y = ma_y$$

$$-mg - bv = m\frac{dv}{dt}$$

Bring the terms with v to the same side.

$$m\frac{dv}{dt} + bv = -mg$$

Divide both sides by m.

$$\frac{dv}{dt} + \frac{b}{m}v = -g$$

This is a first-order linear inhomogeneous ODE, so it can be solved by using an integrating factor.

$$I = \exp\left(\int^t \frac{b}{m}\,dt'\right) = e^{bt/m}$$

Multiply both sides of the ODE by I.

$$e^{bt/m}\frac{dv}{dt} + \frac{b}{m}e^{bt/m}v = -ge^{bt/m}$$

The left side can be written as $\frac{d}{dt}(Iv)$ by the product rule.

$$\frac{d}{dt}(e^{bt/m}v) = -ge^{bt/m}$$

Integrate both sides with respect to t.

$$e^{bt/m}v = -g\frac{m}{b}e^{bt/m} + C_1$$

Divide both sides by I.

$$v(t) = -\frac{mg}{b} + C_1e^{-bt/m}$$

Use the fact that the initial velocity is $v(0) = v_\text{o}$ to determine C_1.

$$v(0) = -\frac{mg}{b} + C_1 = v_\text{o} \quad \rightarrow \quad C_1 = v_\text{o} + \frac{mg}{b}$$

Therefore, the object's velocity is

$$v(t) = -\frac{mg}{b} + \left(v_\text{o} + \frac{mg}{b}\right)e^{-bt/m}$$

$$\boxed{v(t) = -v_\text{ter} + (v_\text{o} + v_\text{ter})e^{-t/\tau}.}$$

Integrate this result to get the position.

$$\begin{aligned} y(t) &= \int v(t)\,dt \\ &= -\frac{mg}{b}t + \left(v_\text{o} + \frac{mg}{b}\right)\left(-\frac{m}{b}\right)e^{-bt/m} + C_2 \end{aligned}$$

Since y is measured upward from the point of release, $y = 0$ at $t = 0$. Use this fact to determine C_2.

$$y(0) = \left(v_\text{o} + \frac{mg}{b}\right)\left(-\frac{m}{b}\right) + C_2 = 0 \quad \rightarrow \quad C_2 = \frac{m}{b}\left(v_\text{o} + \frac{mg}{b}\right)$$

Therefore, the object's position is

$$\begin{aligned} y(t) &= -\frac{mg}{b}t + \left(v_\text{o} + \frac{mg}{b}\right)\left(-\frac{m}{b}\right)e^{-bt/m} + \frac{m}{b}\left(v_\text{o} + \frac{mg}{b}\right) \\ &= -\frac{mg}{b}t + \frac{m}{b}\left(v_\text{o} + \frac{mg}{b}\right)(1 - e^{-bt/m}) \end{aligned}$$

$$\boxed{y(t) = -v_\text{ter}t + \tau(v_\text{o} + v_\text{ter})(1 - e^{-t/\tau}).}$$

Part (b)

The object reaches its highest point when it comes to a stop, so set $v(t_{\text{max}}) = 0$ and solve the equation for t_{max} to find when this happens.

$$0 = -\frac{mg}{b} + \left(v_{\text{o}} + \frac{mg}{b}\right) e^{-bt_{\text{max}}/m}$$

$$\frac{mg}{b} = \left(v_{\text{o}} + \frac{mg}{b}\right) e^{-bt_{\text{max}}/m}$$

$$\frac{\frac{mg}{b}}{v_{\text{o}} + \frac{mg}{b}} = e^{-bt_{\text{max}}/m}$$

$$\frac{mg}{bv_{\text{o}} + mg} = e^{-bt_{\text{max}}/m}$$

$$\ln \frac{mg}{bv_{\text{o}} + mg} = \ln e^{-bt_{\text{max}}/m}$$

$$\ln \frac{mg}{bv_{\text{o}} + mg} = -\frac{bt_{\text{max}}}{m} \ln e$$

$$\ln \frac{bv_{\text{o}} + mg}{mg} = \frac{bt_{\text{max}}}{m}$$

Therefore, the time when the object reaches its highest point is

$$t_{\text{max}} = \frac{m}{b} \ln \left(\frac{bv_{\text{o}}}{mg} + 1\right)$$

$$\boxed{t_{\text{max}} = \tau \ln \left(\frac{v_{\text{o}}}{v_{\text{ter}}} + 1\right).}$$

Plug this into the position function to determine y_{max}.

$$\begin{aligned}
y_{\text{max}} &= y(t_{\text{max}}) \\
&= -\frac{mg}{b} t_{\text{max}} + \frac{m}{b}\left(v_{\text{o}} + \frac{mg}{b}\right)(1 - e^{-bt_{\text{max}}/m}) \\
&= -\frac{mg}{b}\frac{m}{b} \ln \left(\frac{bv_{\text{o}}}{mg} + 1\right) + \frac{m}{b}\left(v_{\text{o}} + \frac{mg}{b}\right)\left[1 - \exp\left(-\ln \frac{bv_{\text{o}} + mg}{mg}\right)\right] \\
&= -\frac{mg}{b}\frac{m}{b} \ln \left(\frac{bv_{\text{o}}}{mg} + 1\right) + \frac{m}{b}\left(v_{\text{o}} + \frac{mg}{b}\right)\left[1 - \exp\left(\ln \frac{mg}{bv_{\text{o}} + mg}\right)\right] \\
&= \frac{m}{b}\left[-\frac{mg}{b} \ln \left(\frac{bv_{\text{o}}}{mg} + 1\right) + \left(v_{\text{o}} + \frac{mg}{b}\right)\left(1 - \frac{\frac{mg}{b}}{v_{\text{o}} + \frac{mg}{b}}\right)\right]
\end{aligned}$$

Therefore, the object's maximum height is

$$y_{\text{max}} = \frac{m}{b}\left[-\frac{mg}{b}\ln\left(\frac{bv_{\text{o}}}{mg}+1\right)+\left(v_{\text{o}}+\frac{mg}{b}\right)-\frac{mg}{b}\right]$$

$$= \frac{m}{b}\left[-\frac{mg}{b}\ln\left(\frac{bv_{\text{o}}}{mg}+1\right)+v_{\text{o}}\right]$$

$$\boxed{y_{\text{max}} = \tau\left[-v_{\text{ter}}\ln\left(\frac{v_{\text{o}}}{v_{\text{ter}}}+1\right)+v_{\text{o}}\right].}$$

Part (c)

If the drag coefficient b is very small, then $bv_{\text{o}}/(mg)$ is very small as well. It's then a reasonable approximation to replace $\ln(1+x)$ with the first few terms of its Taylor series expansion about $x=0$, $x - x^2/2$, as the higher-order terms are negligible compared to the first two.

$$y_{\text{max}} = \frac{m}{b}\left[-\frac{mg}{b}\ln\left(\frac{bv_{\text{o}}}{mg}+1\right)+v_{\text{o}}\right]$$

$$\approx \frac{m}{b}\left\{-\frac{mg}{b}\left[\left(\frac{bv_{\text{o}}}{mg}\right)-\frac{1}{2}\left(\frac{bv_{\text{o}}}{mg}\right)^2\right]+v_{\text{o}}\right\}$$

$$\approx \frac{m}{b}\left(-v_{\text{o}}+\frac{bv_{\text{o}}^2}{2mg}+v_{\text{o}}\right)$$

$$\approx \frac{v_{\text{o}}^2}{2g}$$